Q.No.- State and Prove Taylor's form of mean value theorem.

Dr. Sunil Je..
S.B. College

Statement:- If a function $f(x)$ defined in the closed interval $[a, a+h]$ is such that,

(i) $f^{n-1}(x)$ exists and is continuous in $[a, a+h]$.

(ii) $f^{n-1}(x)$ Possesses a derivative, in open interval $]a, a+h[$.

(iii) p is any +ve number such that $0 < p \leq n$,

then there exists at least one number θ such that,

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{\lfloor 2}f''(a) + \cdots + \frac{h^{n-1}}{\lfloor(n-1)}f^{n-1}(a)$$
$$+ \frac{h^n(1-\theta)^{n-p}}{\lfloor(n-1)\, p}f^n(a+\theta h)$$

Where, $0 < \theta < 1$.

Proof:- To Prove this theorem, let us first observe that Condition (i) in the statement of the theorem implies that $f, f', f'', \cdots f^{n-1}$ are all defined and continuous in $[a, a+h]$.

Consider the auxiliary function.

$$\phi(x) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{\lfloor 2}f''(x) + \cdots$$
$$\cdots + \frac{(a+h-x)^{n-1}}{\lfloor(n-1)}f^{n-1}(x) + (a+h-x)^p \cdot A \quad \text{—— (1)}$$

Where A is a constant so chosen that,

$$\phi(a) = \phi(a+h) \quad \text{—— (2)}$$

Now, from (1)

$$\phi(a) = f(a) + hf'(a) + \frac{h^2}{\lfloor 2}f''(a) + \cdots + \frac{h^{n-1}}{\lfloor(n-1)}f^{n-1}(a)$$
$$+ Ah^p;$$

and $\phi(a+h)=f(a+h)$.

Therefore, from (2), it follows that,

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{\lfloor 2}f''(a)+\cdots+\frac{h^{n-1}}{\lfloor(n-1)}f^{n-1}(a)$$

$$+Ah^p \quad \text{--- (3)}$$

We now observe that (1) that (i) $\phi(x)$ is continuous in $[a, a+h]$, since by hypothesis $f(x), f'(x), \cdots f^{n-1}(x)$ are all continuous in $[a, a+h]$ and also $(a+h-x)^r$, $r=1,2,3,\cdots n-1, p$ is continuous in $[a, a+h]$.

(ii) $\phi(x)$ is derivable in $]a, a+h[$

and (iii) $\phi(a)=\phi(a+h)$.

Hence $\phi(x)$ satisfies all the three conditions of Rolle's theorem and consequently there exists at least one $\xi \in]a, a+h[$ such that $\phi'(\xi)=0$.

Now, we calculate $\phi'(x)$ as follows:

From (1) differentiating both sides w.r.t. x, we have

$$\phi'(x)=f'(x)+\{(a+h-x)f''(x)-f'(x)\}$$

$$+\left\{\frac{(a+h-x)^2}{\lfloor 2}f'''(x)-2\frac{(a+h-x)}{2}f''(x)\right\}$$

$$+\left\{\frac{(a+h-x)^3}{\lfloor 3}f''''(x)-\frac{3(a+h-x)^2}{\lfloor 3}f'''(x)\right\}$$

$$+\cdots\cdots+\left\{\frac{(a+h-x)^{n-1}}{\lfloor(n-1)}f^n(x)\right.$$

$$\left.-\frac{(n-1)(a+h-x)^{n-2}}{\lfloor(n-1)}f^{n-1}(x)\right\}$$

$$+A\cdot p(a+h-x)^{p-1}(-1)$$

$$=\frac{(a+h-x)^{n-1}}{\lfloor(n-1)}f^n(x)-pA(a+h-x)^{p-1}$$

$$\therefore \phi'(\xi)=\frac{(a+h-\xi)^{n-1}}{\lfloor(n-1)}f^n(\xi)-pA(a+h-\xi)^{p-1}.$$

But $\phi'(\xi) = 0$, therefore,

$$p A(a+h-\xi)^{p-1} = \frac{(a+h-\xi)^{n-1}}{\lfloor(n-1)} f^{n}(\xi),$$

$$\Rightarrow pA = \frac{(a+h-\xi)^{n-p}}{\lfloor(n-1)} f^{n}(\xi)$$

$$\therefore A = \frac{(a+h-\xi)^{n-p}}{\lfloor(n-1)\, p} f^{n}(\xi),$$

If we write $\xi = a + \theta h$ where, $0 < \theta < 1$, then

$$A = \frac{(a+h-a-\theta h)^{n-p}}{\lfloor(n-1)\, p} f^{n}(a+\theta h).$$

$$= \frac{h^{n-p}(1-\theta)^{n-p}}{\lfloor(n-1)\, p} f^{n}(a+\theta h).$$

Now, Putting the value of A in (3), we have

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{\lfloor 2} f''(a) + \frac{h^3}{\lfloor 3} f'''(a) + \cdots$$

$$\cdots + \frac{h^{n-1}}{\lfloor(n-1)} f^{n-1}(a) + \frac{h^{n}(1-\theta)^{n-p}}{\lfloor(n-1)\, p} f^{n}(a+\theta h).$$

which Proves the theorem.